Algorithmic Graph Theory

Summary based on the lecture by Eranda Dragoti-Cela in the summer term 2009

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1 Introduction, Basic Notions and Notations

Definition (graph)**.** $G = (V, E)$. V: *node set.* $E \subseteq V \times V$: *edge set.*

Definition. A set, $p \in \mathbb{N}_0$. $\binom{A}{p} = \{B \subseteq A : |B| = p\}$.

Definition. *Simple* graphs: no multiple edges and no loops.
$n := |V|$: *order* of G. $m := |E|$: *size* of G.
G is *even* (*odd*) if n is even (odd).
If $e = \{u, v\}$, then u, $v \in V$ are *endpoints* of e.
$v \in V$ and $e \in E$ are *incident* if v is an endpoint of e.
$\overline{G} = \left(V, \binom{V}{2} \setminus E\right)$ is the *complement* of G.

Remark. There are $2^{\binom{n}{2}}$ different graphs on n different nodes.

Definition. $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ are *isomorphic* if there is a bijection $f : V_1 \to V_2$ s.t. $(u, v) \in E_1$ iff $(f(u), f(v)) \in E_2$.

Definition. The *neighborhood* of v is

$$\Gamma(v) = \{u \in V : (u, v) \in E\}.$$

The *degree* of v is $\mathrm{d}_G(v) = |\Gamma(v)|$.

Remark ("Handschlaglemma")*.* $\sum_{v \in V} \mathrm{d}_G(v) = 2m$.

Definition. $G = (V, E)$ is *r-regular* if $\mathrm{d}_G(v) = r$ for all $v \in V$.
Maximum degree: $\Delta(G) = \max\{\mathrm{d}_G(v) : v \in V\}$.
Minimum degree: $\delta(G) = \min\{\mathrm{d}_G(v) : v \in V\}$.

Definition. *Adjacency matrix* of G: $A = (a_{ij})_{ij} \in \{0, 1\}^{n \times n}$,

$$a_{ij} = \begin{cases} 1 & (v_i, v_j) \in E \\ 0 & \text{otherwise} \end{cases}$$

Definition. $H = (V_H, E_H)$ is a *subgraph* of $G = (V, E)$ if there is an injection $f : V_H \to V$ s.t. $\{u, v\} \in E_H$ implies $\{f(u), f(v)\} \in E$.
$H \subseteq G$ is an *induced subgraph* if $\exists S \subseteq V$ s.t. $H = \left(S, \binom{S}{2} \cap E\right)$. Denoted by $G[S]$.
$H \subseteq G$ is *spanning* if $V_H = V$.

Definition. $C \subseteq V$ is a *clique* if $G[C]$ is complete.
$S \subseteq V$ is *stable* if $G[S]$ is empty.
The maximal cardinality of a clique in G is called *clique number* $\omega(G)$.
The maximal cardinality of a stable set in G is called *independent number* $\alpha(G)$.

Remark. $\alpha(G) = \omega(\overline{G})$.

Remark. $1 \le \omega(G) \le \Delta(G) + 1$.

Remark. $\alpha(G) \ge \frac{n}{1+\Delta(G)}$.

1.1 Walks, paths and connectedness

Definition. A *walk* is a sequence of nodes $(v_1, v_2, \dots, v_l)$ s.t. $(v_i, v_{i+1}) \in E$.
A walk with pairwise distinguished nodes is a *path* (a (v_1, v_l)-path).
A path where the first and last node coincide is a *cycle*.

Definition.

$$\mathrm{dist}(u,v) := \begin{cases} \#\text{ edges of the shortest } (u,v)\text{-path}, & \text{if any exists,} \\ \infty & \text{otherwise} \end{cases}$$

Definition. G is *connected* if $d(u,v) < \infty$ for all u, $v \in V$.
An inclusion-maximal connected subgraph in G is a *connected component* of G.
The number of connected components in G is called *connected number* $c(G)$.

Remark. Equivalent definition: equivalent classes of equivalence relation R with

$$u \, R \, v \Leftrightarrow d(u,v) < \infty.$$

Proposition 1.1 (Dirac 1952)**.**

1. *Every graph G with $\delta(G) \geq 2$ contains a cycle of length $\geq \delta(G) + 1$.*
2. *If $d = \frac{m}{n} \geq 2$ then G contains a cycle of length $\geq d$.*

Definition. The *girth* $g(G)$ is the length of a shortest cycle in G. If there are no cycles in G, then $g(G) = \infty$.

Proposition 1.2. *For a graph G with $\delta(G) = \frac{n}{2}$ it holds that $g(G) = 3$.*

1.2 Datastructure representation for graphs

1.2.1 Static representations

- adjacency matrix
- list of edges
- incidence matrix

1.2.2 Dynamic representations

- adjacency list

1.3 Algorithmic principles

1.3.1 Breadth-first search

Definition.

$$S_i(u) = \{v \in V : \operatorname{dist}(u, v) = i\}$$

is the *i-th sphere around u*.

BFS determines $S_i(u)$ consecutively starting with $i = 0$ and then constructs $S_{i+1}(u)$ by considering neighbors of vertices in $S_i(u)$.

Proposition 1.3. *BFS runs in $\mathcal{O}(m + n)$ time.*

Definition. The *recognition problem* for a class $\mathcal{G}$ of graphs is: Given a graph G, does G belong to $\mathcal{G}$?

G is *recognizable in polynomical time* if for every graph G the question "$G \in \mathcal{G}$?" can be answered in polynomial time.

Proposition 1.4.

1. *$\operatorname{dist}(u, v)$ for all u, $v \in V$ can be computed in $\mathcal{O}(n(n + m))$ time.*
2. *Connectedness can be checked in linear time.*
3. *The connected component of a graph can be determined in linear time $\mathcal{O}(n + m)$.*
4. *A spanning tree can be determined in linear time.*
5. *Checking if a graph is cycle-free can be done in linear time.*
6. *Trees can be recognized in linear time.*

Proposition 1.5. *Bipartite graphs can be recognized in $\mathcal{O}(n + m)$ time.*

Proof. G bipartite $\Leftrightarrow V_1 = \{v \in V : \operatorname{dist}(u, v) = 0 \mod 2\}$, $V_2 = \{v \in V : \operatorname{dist}(u, v) = 1 \mod 2\}$ is a valid partition of G. □

1.3.2 Depth-first search

Proposition 1.6.

1. *$\{\{\operatorname{vor}[v], v\} : v \in V \setminus \{s\}\}$ is a spanning tree.*
2. *DFS runs in $\mathcal{O}(n + m)$ time.*

2 Durchlaufbarkeit

2.1 Eulerian graphs

Definition. G is called *Eulerian* if there exists a closed walk containing all edges of G exactly once.

Theorem 2.1 (Euler 1736, Hierholzer 1972, Veblen 1912)**.** *$G = (V, E)$ connected. The following are equivalent:*

1. *G is Eulerian.*
2. *$\mathrm{d}_G(v)$ is even for all $v \in V$.*
3. *E can be partitioned in disjoin cycles.*

Proof. $1 \Rightarrow 2$: trivial.
$2 \Rightarrow 3$: iteratively remove cycle C (exists because no leaves $\Rightarrow$ G not a forest) from G.
$3 \Rightarrow 1$: induction on number of cycles; every connected component of $G_1 = C_1 \cup C_2 \cup \cdots \cup C_{k-1}$ is Eulerian by induction, adding C_k makes it connected. □

Hierholzer algorithm: Eulerian walk coded by single linked list. As long as there are edges adjacent to current position, append a new cycle.

Proposition 2.2. *The algorithm Hierholzer constructs a Eulerian tour in a connected Eulerian graph in $\mathcal{O}(m)$ time.*

2.2 Hamiltonian graphs

Definition. A *Hamiltonian path* (*cycle*) in G is a path (cycle) containing every node of G. G is *Hamiltonian* if it contains a Hamiltonian cycle.

Lemma 2.3. *G Hamiltonian. For all non-empty $S \subset V$: $G - S$ has at most $|S|$ connected components.*

Theorem 2.4 (Karp 1972)**.** *The problem* Hamiltonian Cycle *(does there exist a Hamiltonian cycle?) is NP-complete.*

Corollary. The problem *Circumference* (determine a longest cycle) is NP-complete.

Many sufficient conditions for Hamiltonicity require something about the density of G or $\delta(G)$. But a "sparse" graph with $\delta(G) = 2$ can also be Hamiltonian, e.g. C_n.

Lemma 2.5. *Let u, $v \in V$ s.t. $\{u, v\} \notin E$ and $\mathrm{d}(u) + \mathrm{d}(v) \geq n$. Then G is Hamiltonian iff $G + \{u, v\}$ is Hamiltonian.*

Proof. $\Rightarrow$: trivial.
$\Leftarrow$: do "pair-exchange" in Hamiltonian cycle that contains $\{u, v\}$. □

Definition. The *k-th Hamiltonian hull* of G, $H_k(G)$, is recursively defined as $H_k(G) = H_k(G + \{u, v\})$ if there are $u, v \in V : \{u, v\} \notin E$ and $\mathrm{d}(u) + \mathrm{d}(v) \geq k$, and $H_k(G) = G$ otherwise.

Lemma 2.6. *The k-th Hamiltonian hull is properly defined.*

Theorem 2.7 (Bondy, Chvátal 1976)**.** *G is Hamiltonian iff its n-th Hamiltonian hull is Hamiltonian.*

Corollary (Ore 1960)**.** If $|V| \geq 3$ and

$$\forall u, v \in V : \{u, v\} \notin E \to \mathrm{d}(u) + \mathrm{d}(v) \geq n \tag{1}$$

("condition of Ore") then G is Hamiltonian.

Corollary (Dirac 1952)**.** If $|V| \geq 3$ and $\delta(G) \geq \frac{n}{2}$ then G is Hamiltonian.

Definition. The *connection number* $\kappa(G)$ is defined as

$$\kappa(G) := \max\{k \in \mathbb{N} : \forall u, v \in V, u \neq v, \exists \text{ at least } k \text{ node-disjoint } (u, v)\text{-paths}\}.$$

Lemma 2.8. *$|V| \geq k + 1$, G not complete. Then the following are equivalent:*

1. *$\kappa(G) = k$.*
2. *Every separating node set has at least k nodes.*

Theorem 2.9 (Chvátal, Erdős 1972)**.** *If $|V| \geq 3$ and $\kappa(G) \geq \alpha(G)$ then G is Hamiltonian.*

Proof. If $\alpha(G) = 1$, $G = K_n$ is Hamiltonian.

If $\alpha(G) \geq 2$, let C be a cycle in G with $|C| < n$. Let S be a connected component in $G[V \setminus C]$ and let

$$T = C \cap \left(\bigcup_{s \in S} \Gamma(s) \right).$$

Choose an orientation of C and let v_i^* be the successor of v_i in C. Extend C either

1. by a (v_i, v_i^*)-path through S if $v_i^* \in T$ or
2. by a path v_i—v_j—v_i^*—v_j^* if $\{v_i^*, v_j^*\} \in E$.

This is possible because otherwise

$$\kappa(G) \leq |T| = |T^*| < |T^* \cup \{s\}| \leq \alpha(G).$$

□

Remark. Theorem 2.9 is best possible. Petersen graph ("extended" pentagram) has $\alpha(G) = 4$ and $\kappa(G) = 3$ and is not Hamiltonian.

Corollary. There is an $\mathcal{O}(nm)$ algorithm which determines a Hamiltonian cycle in a graph with $|V| \geq 3$ and $\kappa(G) \geq \alpha(G)$.

Theorem 2.10 (Bondy 1978)**.** *If* $|V| \geq 3$ *and* (1) *then* $\kappa(G) \geq \alpha(G)$.

Proof. Let S be a stable set and A a separating set. Show $|S| \leq |A|$. Let X be a connected component of $G - A$ s.t. $X \cap S \neq \emptyset$. Let $x \in X \cap S$ and $Y = V \setminus (A \cup X) \neq \emptyset$.

1. $S \cap Y = \emptyset$: $|S| \leq |A| - 1$.
2. $S \cap Y \neq \emptyset$: $|S| \leq |A|$.

□

Remark. If G fulfills (1) there is an ordering of nodes $v_1, \dots, v_n$ s.t. $G[\{v_1, \dots, v_n\}]$ is Hamiltonian unless G is isomorphic to $K_{\frac{n}{2},\frac{n}{2}}$ or $K_{\frac{n}{2}} \times K_2$.

Theorem 2.11 (Ore 1960, Bermond 1976, Linial 1976)**.** *Let* G *be 2-connected s.t.*

$$\forall u, v \in V : \{u, v\} \notin E \rightarrow \mathrm{d}(u) + \mathrm{d}(v) \geq d.$$

Then G *contains a cycle of length* $\geq \min\{n, d\}$.

Proof.

1. Construction of a "long" path in G.
2. Construction of a cycle with length $\geq \min\{d, n\}$.

□

Remark. 2-connectedness is necessary: Consider two $K_{\frac{n+1}{2}}$ joined at an articulation c.

Corollary. Let G be connected and

$$d = \min\{\mathrm{d}(u) + \mathrm{d}(v) : \{u, v\} \notin E\}.$$

Then $d > 2\delta(G)$ and G contains a path of length $\geq \min\{n - 1, d\}$.

Proposition 2.12. *If* $\kappa(G) \geq 2$ *and*

$$\mathrm{d}(u) + \mathrm{d}(v) + \mathrm{d}(w) \geq n + \kappa(G)$$

for all stable triples $\{u, v, w\} \subset V$ *then* G *is Hamiltonian.*

Proposition 2.13. *If* G *is* k*-connected and*

$$\left| \bigcup_{s \in S} \Gamma(s) \right| > \frac{k}{k+1}(n-1)$$

for all stables $S \in \binom{V}{k}$ *then* G *is Hamiltonian.*

Proposition 2.14. *Let* $\alpha_{u,v}$ *be the cardinality of a largest stable containing* u *and* v*, and* $\Gamma_{u,v} = |\Gamma(u) \cap \Gamma(v)|$. *If*

$$\{u, v\} \notin E \rightarrow \alpha_{u,v} \geq \Gamma_{u,v},$$

then G *is Hamiltonian.*

3 Graph Coloring

Definition. A *(feasible) k-coloring* in G is $c : V \to \{1, 2, \ldots, k\}$ s.t. $\{u, v\} \in E \to c(u) \neq c(v)$.

G is called *k-colorable* if a k-coloring exists.

The *chromatic number* $\chi(G)$ is the smallest $k \in \mathbb{N}$ s.t. G is k-colorable.

Remark. $\chi(G) \geq \omega(G)$.

G bipartite iff G 2-colorable.

$\chi(G) \geq \frac{n}{\alpha(G)}$.

Example.

$$\chi(C_n) = \begin{cases} 2 & n \text{ even} \\ 3 & n \text{ odd} \end{cases} \qquad \omega(C_n) = \begin{cases} 2 & n \geq 4 \\ 3 & n = 3 \end{cases}$$

Example. $\alpha(K_n) = 1$, $\chi(K_n) = n$.

3.1 Greedy heuristics for graph coloring

```
function Greedy-color(G, σ : {1, 2, ..., n} → V)
    c(v_1) = 1
    for i = 2, ..., n do
        c(v_i) = min{k ∈ ℕ : k ≠ c(u) ∀u ∈ Γ(v_i)}
    end for
    return |{c(v) : v ∈ V}|
end function
```

Proposition 3.1. *$\chi(G) \leq \mathrm{GC}(G, \sigma) \leq \Delta(G) + 1$.*

$\mathrm{GC}(G, \sigma)$ can be computed in $\mathcal{O}(n + m)$ time.

Remark. Equality for odd cycles and complete graphs.

For $K_{1,n}$ (stars), $\chi(K_{1,n}) = \mathrm{GC}(K_{1,n}, \sigma) = 2$, but $\Delta(K_{1,n}) = n + 1$.

Remark. Consider $B_n = (V_n^{(1)} \uplus V_n^{(2)}, V_n^{(1)} \times V_n^{(2)} - \text{(perfect matching)})$: $\chi(B_n) = 2$, $\Delta(B_n) + 1 = \mathrm{GC}(B_n, \sigma) = n$ for $\sigma = (v_1, u_1, v_2, u_2, \ldots, v_n, u_n)$.

$$\frac{\mathrm{GC}(B_n, \sigma) - \chi(B_n)}{\chi(B_n)} = O(n) \to \infty.$$

Theorem 3.2 (Kucera 1991)**.** *Let ε, $\delta > 0$ and $0 < c < 1$ be constants. Then there exists $n_0 \in \mathbb{N}$ s.t. for all $n \geq n_0$ there is a graph G_n with $V(G_n) \geq n$, $\chi(G_n) \leq n^{\varepsilon}$ and the proportion of node orderings for which* GC *uses less than $c\frac{n}{\log n}$ colors is $\mathcal{O}(n^{-\delta})$.*

$$\frac{\mathrm{GC}(G_n, \sigma)}{\chi(G_n)} \geq \frac{c\frac{n}{\log n}}{\chi(G_n)} \to \infty \quad \textit{for } n \to \infty$$

for $1 - o(n^{-\delta})$ orderings.

Lemma 3.3. *For all graphs G there is an ordering σ^* s.t.* $\mathrm{GC}(G, \sigma^*) = \chi(G)$.

```
function SMALLEST-LAST(G)
    for i = n, ..., 1 do
        Choose a node v_i with minimal degree in G
        G = G − v_i
    end for
    return (v_i)_i
end function
```

Proposition 3.4 (Helin 1967, Matula 1968, Finch & Sachs 1969)**.**

$$b(\sigma_{\mathrm{SL}}) = \max_H \delta(H) = \min_{\delta \in S_n} b(\sigma)$$

where $b(\sigma) = \max_{1 \le i \le n} \mathrm{d}_{G_i}(v_i)$.

Corollary. $\chi(G) \le \mathrm{GC}(G, \sigma_{\mathrm{SL}}) \le 1 + \max_H \delta(H)$.

Corollary. If G connected and not $\Delta(G)$-regular, then $\chi(G) \le \Delta(G)$.

Proof. Let H_0 be subgraph with maximal $\delta(H)$ and $\delta(H_0) = \Delta(G)$. If $V(H_0) = V$, then $H_0 = G$, contradiction to G not $\Delta(G)$-regular. If $V(H_0) \subset V$, then $\{v, x\} \notin E$, contradiction to G connected. □

Algorithmic proof. Let $v_1 \in V$ s.t. $\mathrm{d}(v_1) < \Delta(G)$. Do $\mathrm{BFS}(v_1)$ and number nodes accordingly. In every step i, GC uses as many colors as there are colored neighbors of $v_1 + 1$. There are $\le \Delta(G) - 1$ such neighbors because at least one neighbor with smaller index than v_i is not colored yet. □

3.2 Two-connectivity and the block structure

Theorem 3.5 (Partitioning in "ears"; Whitney 1932, Halin, Jung 1963)**.** *G is 2-connected iff it can be presented as $G = C \cup P_1 \cup P_2 \cup \cdots \cup P_k$, where C is a cycle and P_i are paths s.t. P_i and $C \cup P_1 \cup \cdots \cup P_{k-1}$ have only the end nodes of P_i in common.*

Proof. $\Leftarrow$: induction on k, construct two paths to $s \in G_{k-1}$, $t \in P_k - \{a_1, a_2\}$ through a_1, a_2.

$\Rightarrow$: Assume $G_{k-1} \neq G$. Construct P_k using edge $e = \{u, v\} \in E \setminus E(G_{k-1})$ with $v \in G_{k-1}$ and (if necessary) path the connects u to $V(G) - v$. □

Theorem 3.6. *If G has $|V| \ge 3$ and no isolated nodes, then the following are equivalent:*

1. *G is 2-connected.*
2. *$\forall e_1, e_2 \in E \;\exists$ cycle $C : e_1, e_2 \in C$.*
3. *$\forall e \in E \;\forall v \in V \;\exists$ cycle $C : e, v \in C$.*

4. $\forall v_1, v_2 \in V \; \exists$ *cycle* $C : v_1, v_2 \in C$.

Definition. A *block* is an inclusion-maximal subgraph which is connected and has no articulation points.

Remark. Blocks are either edges or 2-connected subgraphs of G.

Lemma 3.7. *A cycle C is contained in one block of G.*

Lemma 3.8. *The blocks of G form a partition of its edge set E.*

Proof. Assume $e \in E(B_1) \cap E(B_2)$, $f_1 \in E(B_1)$, $f_1 \notin E(B_2)$, $f_2 \in E(B_2)$, $f_1 \notin E(B_1)$. Consider cycle C that contains e and f_1 (Theorem 3.6): B_2 could be extended by $C - e$. □

Lemma 3.9 (König 1936)**.** *Two different blocks B_1 and B_2 of G have at most one node in common; $v \in V$ is contained in more than one block iff v is an articulation.*

Definition (block-articulation graph)**.**

$$V(\mathrm{bc}(G)) = \mathcal{B} \cup \mathcal{A} \qquad \text{(blocks and articulations)}$$
$$E(\mathrm{bc}(G)) = \{\{B, a\} : B \in \mathcal{B}, a \in \mathcal{A}\}$$

Theorem 3.10 (Galloi 1964, Harary & Prins 1966)**.** $\mathrm{bc}(G)$ *is a tree.*

Proof. Paths in G imply paths in $\mathrm{bc}(G)$. Cycle in $\mathrm{bc}(G)$ would be contained in two different blocks. □

Theorem 3.11 (Brooks 1941)**.** *Let G be connected with $|V| \geq 3$. If G is not complete and not an odd cycle, then $\chi(G) \leq \Delta(G)$.*

Proof. Let B be a block in $\mathrm{bc}(G)$. If B is regular, then it is not $\Delta(G)$-regular, therefore $\mathrm{GC}(B, \sigma) \leq 1 + \Delta(B) \leq \Delta(G)$. Otherwise it can be colored with at most $\Delta(B) \leq \Delta(G)$ colors by Corollary 2 to Proposition 3.4. Color every block and permute colors s.t. no collisions at block-joining edges/nodes.

Assume w.l.o.g. that G is not a cycle. Choose $v_1 \in V$ with neighbors v_n, v_{n-1} s.t. $\{v_n, v_{n-1}\} \notin E$ and $G - v_n - v_{n-1}$ is connected (by Lemma 3.12). Do $\mathrm{BFS}(G - v_n - v_{n-1}, v_1)$. Then $\mathrm{GC}(G, \sigma = (v_n, v_{n-1}, \ldots, v_1)) \leq \Delta(G)$: v_n and v_{n-1} get color 1, for others $\mathrm{d}(v_i) \leq \Delta(G)$ and at least one neighbor not yet colored. □

Lemma 3.12. *If G is 2-connected, not complete and not a cycle, then*

$$\exists x, y \in V : \mathrm{dist}(x, y) = 2 \text{ and } G - x - y \text{ is connected.}$$

Corollary. If G is connected, not complete and not an odd cycle, then a coloring which uses $\leq \Delta(G)$ colors can be computed in linear time.

3.3 Chromatic number and girth of a graph

We know $\chi(G) \geq \omega(G)$. The approximation gap can be as large as possible: Consider the Mycielski family of graphs M_k, where $\chi(M_k) = k$ and M_k is triangle-free, i.e. $\omega(M_k) = 2$. $M_3 = C_5$; M_4: Grötzsch graph.

Proposition 3.13 (Mycielski 1955). *M_k is triangle-free, has $3 \times 2^{k-2} - 1$ nodes and $\chi(M_k) = k$.*

Theorem 3.14 (Erdős 1959). *For all k, $k \in \mathbb{N}$, $k, g \geq 3$ there is a graph G s.t. $\chi(G) \geq k$ and $g(G) \geq g$.*

Remark. The girth $g(G)$ is a "local" property, while the chromatic number $\chi(G)$ is a "global" property.

3.4 Coloring algorithms

3.4.1 Exact coloring algorithms

1. Run GC for every permutation σ in time $\Omega(n \cdot n!) = \Omega\left(n^{\frac{3}{2}} \left(\frac{n}{e}\right)^n\right)$.

2. Let $e = \{u, v\} \notin E$. Work with $G + e$ and G/e.

   Lemma 3.15 (Zykov). *Let $e = \{u, v\} \notin E$.*

   $$\chi(G) = \min\{\chi(G + e), \chi(G/e)\}.$$

   Proof. $\leq$: A coloring of $G + e$ is also a coloring of G, every coloring of G/e implies a coloring of G with $c(u) = c(v)$.

   $\geq$: Let c be an optimal coloring. If $c(u) \neq c(v)$ then c is a coloring for $G + e$, otherwise c is a coloring for G/e. □

   Algorithm constructs a binary tree with root at G and children $H + e$, H/e for every node (graph) H. Time complexity

   $$f(n) \leq \frac{n^2}{2} + \frac{n^2}{2} f(n-1) \leq (n!)^2 f(1) = \mathcal{O}\left((n!)^2\right).$$

3. Independent-set approach of Christofides (1971):

   $$\chi(G) = 1 + \min\{\chi(G - S) : S \text{ max. independent set in } G\}.$$

   for $k = 1, \ldots, n$ **do**
       for all $U \in \binom{V}{k}$ **do**
           $\chi(G[U]) = 1 + \min\{\chi(G[U \setminus S]) : S \text{ max. independent set in } G[U]\}$
       end for
   end for

List all max. indep. sets in $\mathcal{O}(mnk)$ time (k is the number of max. indep. sets), of which there are at most $3^{\frac{n}{3}}$. Time:

$$\sum_{k=1}^{n} \binom{n}{k} \mathcal{O}\left(mk3^{\frac{k}{3}}\right) = \mathcal{O}\left(mn(1+\sqrt[3]{3})^n\right).$$

4. Backtracking according to Brown: best approach known for graphs of small size with around 50 nodes. Idea: repeat greedy coloring by tracking partitions of V in colors sets already produced by previous runs of GC.

3.4.2 Coloring heuristics

1. Saturation-largest-first heuristics (SLF):

   $$\text{dsatur}(v) = |\{c(u) : u \in \Gamma(v) \text{ already colored}\}|.$$

   Select next node v_i to color according to

   $$v_i \in \text{argmax}\{\text{dsatur}(v) : v \in V \text{ not colored yet}\}.$$

   Proposition 3.16. *SLF colors bipartite graphs optimally.*

2. Similarity-merge heuristics (SM): sequentially search for two non-adjacent nodes u, v with maximal number of common neighbors and substitutes them by a supernode $\{u, v\}$, then $G = G/\{u, v\}$.

   Proposition 3.17. *SM colors bipartite graphs optimally.*

   Remark (Kucera 1989)*.* SM colors graphs G with $\chi(G) \leq \sqrt{\frac{n}{1.96 \log n}}$ optimally.

3. Recursive-largest-first heuristics (RLF): color in every step a stable set S with current color, where S is chosen s.t. the cut $[S, U \setminus S]$ is maximized, where U is the set of uncolored nodes at the given moment in time.

   Remark. RLF colors bipartite graphs optimally and can be implemented to run in $\mathcal{O}(nm)$ time.

3.5 Edge coloring: the theorem of Vizing

Definition. The *chromatic index* $\chi'(G)$ is the smallest $n \in \mathbb{N}$ s.t. the edges of G can be colored by k colors where no pair of incident edges share a color.

Remark. $\chi'(G) = \chi(L(G))$, where $L(G)$ is the line graph of G.

Example. $\chi'(C_n) = \begin{cases} 2 & n \text{ even}, \\ 3 & n \text{ odd}. \end{cases}$

Remark. $\chi'(G) \geq \omega(L(G)) \geq \Delta(G)$.

Proposition 3.18. $\chi'(K_n) = \begin{cases} n-1 & n \textit{ even}, \\ n & n \textit{ odd}. \end{cases}$

Theorem 3.19 (Vizing 1964)**.** $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$

Lemma 3.20. *Let $G = (V \cup U, E)$ be bipartite. Then $\chi'(G) = \Delta(G)$ and an optimal edge coloring can be constructed in $\mathcal{O}(nm)$ time.*

Corollary. There is an $\mathcal{O}(nm)$ algorithm to color the edges of a graph G with at most $\Delta(G) + 1$ colors.

4 Topological Graph Theory, Planar Graphs

Some NP-hard problems which become efficiently solvable in planar graphs:

- max cut
- multiterminal cut
- edge-disjoint paths problem
- number of perfect matchings
- approximation of max stable sets in a graph

4.1 Planar graphs

Definition. G is *planar* if there is anembedding of G in the plane such that nodes are embedded into points on the plane, edges are embedded into Jordan curves whose endpoints are nodes, where every Jordan curve intersects the nodes at most at its endpoints, and every pair of Jordan curves intersects also at most at one of their endpoints.

Example. Trees and $K_{2,3}$ are planar.

Remark. There is a homeomorphism between the plane and the surface of a sphere.

Definition. The edges of a planar embedding of a graph divide the plane into connected *regions* s.t. exactly one of them is unbounded (the *outer face*). The *boundary* of a region is defined as the set of edges contained in the (topological) closure of the region.

Remark. Given a planar embedding of a graph and a particular region in the embedding, we can always find a planar embedding for which that particular region becomes the outer face.

Proposition 4.1. *G is planar iff all its blocks (*$\mathrm{bc}(G)$*) are planar.*

Proposition 4.2. *G is 2-connected iff for every planar embedding of G the border of every region is a cycle.*

Definition. Two planar embeddings H, H' with isomorphisms $\varphi : G \to H$, $\varphi' : G \to H'$ are *equivalent* if

$$\varphi' \circ \varphi^{-1} : H \to H'$$

is an isomorphism s.t. for all sequence R of edges in G, $\varphi(R)$ is border of a region in H iff $\varphi'(R)$ is border of a region in H' in the original or in the reverse ordering.

Theorem 4.3 (Wagner, Fáry)**.** *For every embedding of a planar graph in the plane there exists an equivalent embedding with all edges being straight lines except for the outer face edges.*

Definition. A planar graph G is *uniquely embeddable* if all of its planar embeddings are pairwise equivalent.

Example. C_n, P_n, $K_{2,3}$ are uniquely embeddable.

Theorem 4.4 (Wagner). *A 3-connected planar graph is uniquely embeddable.*

Proof. Consider region R of G_1 whose border $b(R)$ is not the border of some region in G_2. $b(R)$ is a cycle C. A *C-bridge* is either a chord in C or a connected component of $G \setminus C$ to which edges joining H and C and their endpoints are added. Let $\varphi : G_1 \to \overline{G}_1$ (planar) be an isomorphism s.t. $\phi(R)$ is an outer region in $\overline{G}_1$. $\overline{G}_2 = \varphi(G_2)$ and $\overline{G}_1$ cannot be equivalent. There must be u, $v \in \overline{V}_2$ inside and outside C, resp., with C-bridges B_u, B_v, which are not separating. Let $\{t_1, \ldots, t_k\} = B_u \cap C$ and consider $\overline{G}_2 \setminus \{t_1, t_k\}$: it is not connected! Contradiction to $\overline{G}_2$ being 3-connected. □

4.2 Theorems of Kuratovski and Wagner

Definition. Let G be obtained from H s.t. every edge of H is substituted by a simple path and paths substituting different edges be inner-node-disjoint. Such a G is called a *subdivision* of H.

Two graphs are called *homeomorphic* if they are both subdivisions of a third graph.

G contains H as a *topological minor* if it contains a subgraph H' which itself is a subdivision of H.

Remark. G is planar iff all subdivisions of G are planar.

Theorem 4.5 (Kuratovski). *G is planar iff K_5 and $K_{3,3}$ are not subdivisions of G.*

Proof. Induction on $|V|$. □

Lemma 4.6. *If G 3-connected with $|V| \geq 5$ then $\exists e \in E$ s.t. G/e is 3-connected.*

Lemma 4.7. *If G/e contains a subdivision of K_5 or $K_{3,3}$, then G also does so.*

Definition. A *convex embedding* of a planar graph G is a planar embedding where all regions are convex, except the outer face, possibly.

Remark. It follows then that inner edges are straight lines.

Proposition 4.8 (Stein 1951, Tutte 1960). *A 3-connected planar graph has a convex embedding.*

Remark. A convex embedding can be found in linear time $\mathcal{O}(n)$.

Definition. G contains H as a *minor* if G contains a subgraph H' which can be transformed into H just by applying a sequence of edge contractions.

Remark. H is a topological minor of $G \Rightarrow H$ is a minor of G.

Proposition 4.9 (Wagner 1937). *A graph G is planar iff it does not contain K_5 or $K_{3,3}$ as a minor.*

4.3 Planarity testing

A planar embedding can be constructed trivially in $\mathcal{O}(n^2)$ time out of planarity testing algorithms. There is also an $\mathcal{O}(n)$ time algorithm for planar embedding. A planar embedding can be uniquely specified by giving $\mathrm{Adj}(v)$ in a fixed order, e.g. in clockwise order.

W.l.o.g. G is 2-connected and $m \leq 3n - 6$ (Eulerian polyhedron formula).

Definition. An *st-numbering* of G is an ordering $v_1, v_2, \ldots, v_n$ of nodes s.t. $\{v_1, v_n\} \in E$ and

$$\forall j \in \{2, \ldots, n-1\}\ \exists i, k : i < j < k \text{ and } \{v_i, v_j\} \in E, \{v_j, v_k\} \in E.$$

Then $s := v_1$ is a *source* and $t := v_n$ is a *sink*.

Definition. Consider a DFS on G. The *low number* of a node v is

$$\mathrm{LOW}(v) := \min\left\{\{v\}, \{w : \exists \{u, v\} \in E : w \text{ is a successor of } v \text{ and } u \text{ is a predecessor of } v\}\right\}.$$

Such an $\{u, v\} \in E$ with $u \to v \to w$ is called *back edge*.

Remark. Equivalent definition:

$$\mathrm{LOW}(v) = \min\left\{\{v\}, \{\mathrm{LOW}(x) : x \text{ son of } v\}, \{w : \{v, w\} \text{ is a back edge}\}\right\}.$$

Theorem 4.10. *The st-numbering algorithm computes correctly an st-numbering for a 2-connected graph.*

Let $G_k = G[\{1, 2, \ldots, k\}]$, where nodes are named (indexed) by their *st*-numbers.

G'_k is G_k with virtual edges and virtual nodes.

Let B_k (the *bush form*) be a planar embedding of G'_k s.t. all virtual nodes lie on its outer face.

Lemma 4.11. *If edge $\{s, t\}$ is drawn in the outer face of G, then all vertices and edges of $G - G_k$ are drawn in the outer face of the partial embedding G_k of G.*

Proof. For all $v \in V(G - G_k)$: $\mathrm{STN}(v) > k$. Therefore there is a (v, t)-path using only nodes u with $\mathrm{STN}(u) > k$. Since t lies on the outer face such a v should also lie on the outer face, otherwise we would have an edge crossing. □

Definition. A *PQ-tree* is a tree consisting of P-nodes, Q-nodes, and leaves. A P-node is a node whose sons can be permuted freely. A Q-node is a node for which the order of its sons can just be reversed.

In the case of B_k, P nodes represent articulations points, Q-nodes represent blocks, and leaves represent virtual nodes.

Lemma 4.12. *There is a sequence of permutations (and reversals) to make all virtual vertices labelled by $k + 1$ occupy a consecutive interval.*

Definition. A leaf labelled $k + 1$ is called *pertinent* to a PQ-tree corresponding to B_k. The *pertinent subtree* is the minimal subtree containing all pertinent leaves.

A node (subtree) of a PQ-tree is *full* if all leaves among its descendants are pertinent.

5 Perfect Graphs

5.1 Introduction

Definition. G is called *perfect* if for all induced subgraphs $G[H]$:

$$\chi(G[H]) = \omega(G[H]).$$

Example. K_n, trees/forests, and bipartite graphs are perfect (each property being *inheritable* w.r.t. induced subgraphs).

Definition. The *clique-partitioning number* is

$$\theta(G) := \min\{k \in \mathbb{N} : V = \biguplus_{i=1}^{k} V_i \text{ where } G[V_i] \text{ is a clique}\}.$$

The *edge-covering number* is

$$\varrho(G) := \min\{|N| : N \subseteq E \text{ where } N \text{ covers all nodes}\}.$$

The *node-covering number* is

$$\tau(G) := \min\{|H| : H \subseteq V \text{ where } H \text{ covers all edges}\}.$$

The *matching number* is

$$\nu(G) := \max\{|M| : M \subseteq E \text{ where } M \text{ is a matching}\}.$$

Remark (Galler 1953). $\tau(G) + \alpha(G) = \varrho(G) + \nu(G) = n$.

Remark (König 1931). If G is bipartite then $\nu(G) = \tau(G)$.

Proposition 5.1. *Complements of bipartite graphs are perfect.*

Proposition 5.2. *Line graphs of bipartite graphs are perfect.*

Theorem 5.3 (Grötschel, Lovész, Schrijver 1981)**.** *The determination of $\chi(G)$, $\alpha(G)$, $\theta(G)$ and $\omega(G)$ is polynomially solvable for perfect graphs.*

Remark. The bipartite graph of which a given graph is a line graph can be determined in linear time.

5.2 The perfect graph theorem and the strong perfect graph theorem

Theorem 5.4 (perfect graph theorem, Lovász 1972)**.** *The following are equivalent:*

1. $\omega(G[H]) = \chi(G[H])$.
2. $\theta(G[H]) = \alpha(G[H])$.
3. $\omega(G[H])\alpha(G[H]) \geq |H|$.

Definition. The *multiplication* $G \circ x$ of G at node x is defined as

$$\begin{aligned} V(G \circ x) &= V \cup \{x'\}, \\ E(G \circ x) &= E \cup \left\{\{x', v\} : v \in V \text{ s.t. } \{x, v\} \in E\right\}, \end{aligned}$$

for an $x' \notin V$.

Let $h = (h_1, \ldots, h_n) \in \mathbb{N}_\infty^n$. Denote by $G \circ h$ a graph with

$$\begin{aligned} V(G \circ h) &= \bigcup_{i=1}^{n} \left\{x_i^{(1)}, \ldots, x_i^{(h_i)}\right\}, \\ E(G \circ h) &= \left\{\{x_i^s, x_j^t\} : \{x_i, x_j\} \in E, 1 \le s \le h_i, 1 \le t \le h_j\right\}. \end{aligned}$$

Lemma 5.5 (Berge 1961)**.** *Let M be obtained from G by node multiplication. Then*

1. *G satisfies 1 $\Rightarrow$ M satisfies 1.*
2. *G satisfies 2 $\Rightarrow$ M satisfies 2.*

Lemma 5.6 (Fulkerson 1971, Lovász 1972)**.** *Let G be a graph whose proper induced subgraphs satisfy 2 and let M be obtained from G by multiplication of vertices. Then if G satisfies 3, also M satisfies 3.*

Corollary. G is perfect iff $\overline{G}$ is perfect.

Remark. C_{2k+1} is not perfect, $\overline{C}_{2k+1}$ is not perfect either. Proper induced subgraphs of C_{2k+1} are perfect (disjoin union of paths). So C_{2k+1} ($\overline{C}_{2k+1}$) is minimally imperfect.

Proposition 5.7 (Strong perfect graph theorem)**.** *The only minimal imperfect graphs are C_{2k+1} and $\overline{C}_{2k+1}$ for $k \ge 2$.*

5.3 Chordal graphs

Definition (Hajnal, Juványi 1958)**.** G is called *chordal* if it does not contain an induced C_k for $k \ge 4$, i.e. every cycle of length ≥ 4 has a chord.

Example. K_n is chordal. A bipartite graph is chordal iff it is a forest.

Lemma 5.8 (Dirac 1961)**.** *G is chordal iff any inclusion-minimal separating set in G is a clique.*

Definition. A node $v \in V$ is called *implicial* if $G[\Gamma(v)]$ is complete.

Lemma 5.9 (Dirac 1961)**.** *Every chordal graph G has a simplicial node. If G is not K_n, then it has at least two non-adjacent simplicial nodes.*

Theorem 5.10 (Berge 1961, Hajnal, Surányi 1958)**.** *Chordal graphs are perfect.*

Proof. Chordality is an inheritable property.

Induction on $\overline{V}$. If G not complete, choose $\{u, v\} \notin E$, i.e. $V \setminus \{u, v\}$ is a separating set. There exists an inclusion-minimal set C. C is a clique by Lemma 5.8. Let H_1, H_2 be connected components of $G \setminus C$. $H_1 \cup C$, $H_2 \cup C$ are perfect by induction assumption. G is obtained from them by identification on clique C, therefore perfect. □

5.3.1 Recognition of chordal graphs

Definition. Let $\sigma : V \to \{1, 2, \ldots, n\}$ be a total ordering of nodes in V. σ is called *perfect (node) elimination scheme* (PES) if every $v \in V$ is simplical in $G[\{u \in V : \sigma(u) \geq \sigma(v)\}]$.

In other words, the *upper neighborhood*

$$\mathrm{UN}(v) := \{u \in \Gamma(v) : \sigma(u) \geq \sigma(v)\}$$

is fully connected.

Proposition 5.11 (Fulkerson, Gross 1965)**.** *G is chordal iff G has a PES σ.*

Proof. $\Rightarrow$: iteratively remove simplicial node v from G.

$\Leftarrow$: let C_k be a cycle in G with $k \geq 4$, and

$$u = \mathrm{argmin}\{\sigma(v) : v \in C_k\}.$$

Consider v, $w \in \Gamma(u) \cap C_k$. Then $\sigma(v) > \sigma(u)$ and $\sigma(w) > \sigma(u)$, therefore v, $w \in \mathrm{UN}(u)$ and $\{v, w\} \in E$. □

Proposition 5.12. *Let G be chordal and let a maximal adjacency search label the nodes $v_1, \ldots, v_n$. Then*

$$\sigma : V \to \{1, \ldots, n\}; \quad \sigma(v_i) = n - i + 1$$

is a PES.

Lemma 5.13. *A total ordering σ on V is a PES iff for all v_i, $v_j \in V$, for which there is a (v_i, v_j)-path P s.t. for all inner nodes u of P: $\sigma(u) < \min\{\sigma(v_i), \sigma(v_j)\}$, $\{v_i, v_j\} \in E$.*

To check whehter a given ordering σ is a PES, we check whether $\mathrm{UN}(v_i)$ is complete without checking the existence of the edge between the same pair of nodes many times.

```
function PES(()G, σ)
    for all v ∈ V do
        A[v] = ∅
    end for
    for i = 1, ..., n do
        u = σ^{-1}(i).
        UN = {w ∈ Γ(u) : σ(w) > σ(u)}
        if A[u] ⊈ UN then
            return false
        end if
        if UN ≠ ∅ then
            f = argmin{σ(w) : w ∈ UN}
            A[f] = A[f] ∪ (UN \ {f})
        end if
    end for
    return true
end function
```

Lemma 5.14. *The algorithm PES can be implemented with time complexity $\mathcal{O}(n + m)$.*